Complex Numbers and Complex Number Arithmetic

1. An important element of ***complex numbers*** is the basic ***imaginary number***:

$$\sqrt{-1} = j$$

(In circuit analysis, we use ***j*** instead of ***i***, as is used in most mathematics books, since ***i*** is reserved for current.)

Note that $j^2 = -1,\ j^3 = -j,\ j^4 = 1,\ etc.$

2. The ***rectangular form*** of a complex number is constructed from the basic imaginary number and ordinary real numbers as:

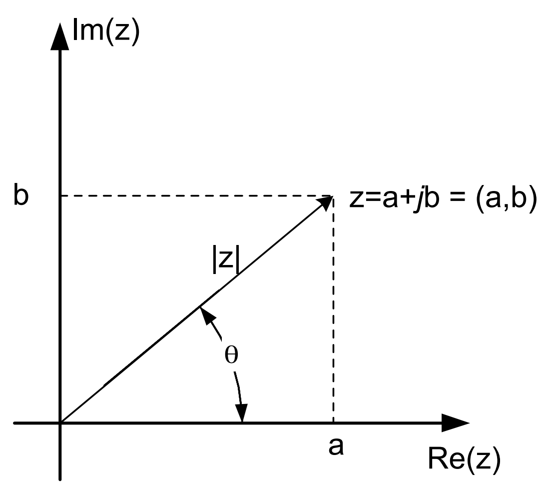
z = a + *j*b,

where ***a*** and ***b*** are the ordinary real numbers, and are respectively the ***real*** and ***imaginary*** parts:

a = Re(z) and
b = Im(z).

3. **Various representation of complex numbers**: Although the rectangular form is the one we will use mainly, complex numbers can be represented in several other ways. These can be related back to the rectangular form by use of the ***complex plane*** as follows:

From this diagram and basic trigonometry, it is clear that:



Complex Plane

$$|z| = \sqrt{a^2+b^2}$$

$$\tan\theta = \frac{b}{a} = \frac{Im(z)}{Re(z)}$$

$a = \cos\theta$

$b = \sin\theta$, and that z can be represented in several equivalent ways:

$$z = a+jb = |z|,\ \theta, = |z|(\cos\theta + j\sin\theta)$$

4. Further, since

$$\cos\theta = \sum 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!}\cdots = \sum_{k=0}^{\infty} \frac{\theta^{2k}}{(2k)!}\left(-1^k\right),$$

$$\sin\theta = \sum \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\cdots = \sum_{k=0}^{\infty} \frac{\theta^{(2k+1}}{(2k+1)!}\left(-1^k\right), \text{ and}$$

$$e^{\theta} = \sum 1 + \theta + \frac{\theta^2}{2!}\cdots = \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \text{ , then}$$

$$z = a + jb = |z|\left(\cos\theta + j\sin\theta\right) = |z|e^{j\theta}$$

Complex Number Arithmetic

4. ***Addition / subtraction:***

$z_1 \pm z_2 = (a + jb) \pm (c + jd) = (a \pm c) + j(b \pm d)$

{Note that the method is like that for adding / subtracting vectors or polynomials; that is, the real parts add together and the imaginary parts add together.}

5. ***Multiplication****:*

$z_1 \times z_2 = (a + jb)x(c + jd) = ac + jbc + jad + j^2bd = (ac - bd) + j(ad + bc)$

{Note again that the method is like that for multiplying polynomials but incorporates the fact that $j^2 = -1$.}

6. ***Complex conjugate: a definition***

For every complex number **z = a + *j*b**, there exists a ***complex conjugate*** defined as **z^* = a - *j*b**. Note that each is the ***conjugate*** of the other, since the complex conjugate is formed by simply ***changing the sign of the imaginary part***.

Note also that:

$z + z^* = (a + jb) + (a - jb) = 2\,\mathrm{Re}(z)$,
$z - z^* = (a + jb) - (a - jb) = 2\,\mathrm{Im}(z)$
$zz^* = (a + jb)^*\,(a - jb) = a^2 + jab - jab + b^2 = a^2 + b^2 = |z|^2$.

7. ***Division:***

The basic strategy is to convert to multiplication as follows:

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}(1) = \frac{z_1}{z_2}\left(\frac{z_2^*}{z_2^*}\right) = \frac{z_1 z_2^*}{|z_2|^2}, \text{ so}$$

$$\frac{a + jb}{c + jd} = \frac{(a + jb)(c - jd)}{c^2 + d^2} = \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2}$$

8. ***Reminder of arithmetic using magnitudes (absolute values):***

Multiplication: $|z_1| \times |z_2| = |z_1 \times z_2|$

Division: $\frac{|z_1|}{|z_2|} = \left|\frac{z_1}{z_2}\right|$

But it is IMPORTANT to note that

Addition / subtraction: $|z_1| \pm |z_2| \neq |z_1 \pm z_2|$!!!!!!